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New Reciprocity Theorems for Chiral, Nonactive, and Biisotropic Media

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Abstract—**Two generalized reciprocity theorems for homogeneous biisotropic media are presented that do not invoke a complementary space. One of them is eminently crosspolarized involving real sources and fields, while the other is a generalization of the Lorentz theorem and is therefore eminently copolarized, invoking generalized sources or fields. These theorems constitute the foundation for new variational expressions leading to a reaction-type development with capabilities to handle biisotropic/nonactive/chiral/isotropic materials.**

I. Introduction

The basic Reciprocity Theorem of Electromagnetics was initially presented by Lorentz [1] for scalar fields, and generalized to vector fields, to become what we know today as the Lorentz form [2], [3]. Lorentz's contribution was a direct extension of the work of Rayleigh in vibrating mechanical systems [4] and optics [5]. Also, as pointed out in [6] and [7], it was Heaviside who, contemporary with the earliest work of Rayleigh, invoked what we know today as reciprocity for Electrical Networks.

Over the years, the basic theorem was extended to suit different circumstances, such as time harmonic fields, time domain, inhomogeneities, boundary conditions, anisotropy, piezoelectric media, and bianisotropy among others. Of special interest to us here is the work of Kong and Cheng [8], who considered the full bianisotropic case and departed from the previous line of thought. In order to state a reciprocity theorem in the vein of the Lorentz Theorem, they invoked a complementary space characterized by material different from that of the original space. This is important to us because biisotropy is a special case of bianisotropy, and in turn, chirality is a special case of biisotropy.

What this means is that a reciprocity theorem is available that applies to both chirality and biisotropy. Whereas chirality is inherently reciprocal (hence the term chiral reciprocal), biisotropy does require the introduction of a (different) complementary space. It is possible, however, to produce a new reciprocity relation for biisotropic and chiral materials by proper exploitation of the technique of decomposition of general fields into circularly polarized components [9], which has been used by this author [7] to obtain a nonLorentzian reciprocal relation for isotropic materials.

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The isotropic work [7] included applications and stressed the fact that the new theorem complements the old one in that the new one is eminently cross-polarized, while the old one is eminently co-polarized. As stated in [7], this concept is important, because it constitutes the foundation for a development of variational expressions that complements Rumsey's Reaction principle [10] and has potential to handle complex systems with high degree of cross-polarization.

The isotropic work in [7] resembles Tai's Complementary Theorem [11]; and it has been recently brought to the attention of the author[1] that one of the main results of [7] was apparently derived almost simultaneously and by entirely independent means by Fel'd and published in the Russian literature under a somewhat misleading title [12].

Aside from the notable work of Kong and Cheng on a reciprocity relation directly applicable to biisotropy, we can also cite the relevant works of Krowne [13] and Lindell *et al.* [14]. Here, we present a new reciprocity theorem that does not require the introduction of a complementary space.

II. Analysis

The constitutive relations for biisotropic media are [9]

$$\overline{D} = \varepsilon \overline{E} + \gamma \overline{H}, \quad \overline{B} = \mu \overline{H} + \beta \overline{E} \tag{1}$$

where the dimensions of γ and β are inverse to that of speed. The medium is lossless if E and μ are real, and $\gamma = \beta^*$. The condition for the medium to be reciprocal is $\gamma = -\beta$, and the resulting material is commonly known as chiral.

A general field decomposition in biisotropic media in terms of RCP/LCP (right/left circularly polarized) fields in the presence of electric ($\overline{J}$) and magnetic ($\overline{M}$) sources is possible via [9], [15]

$$\overline{E} = \overline{E}_+ + \overline{E}_- \quad \overline{H} = \overline{H}_+ + \overline{H}_- \tag{2}$$

$$\overline{J} = \overline{J}_+ + \overline{J}_- \quad \overline{M} = \overline{M}_+ + \overline{M}_- \tag{3}$$

$$\overline{E}_\pm(\overline{r}) = \mp j\eta_\pm \overline{H}_\pm(\overline{r}) \tag{4}$$

$$\eta_\pm = \sqrt{\frac{\mu}{\varepsilon} - \left(\frac{\gamma+\beta}{2\varepsilon}\right)^2} \mp j\left(\frac{\gamma+\beta}{2\varepsilon}\right) \tag{5}$$

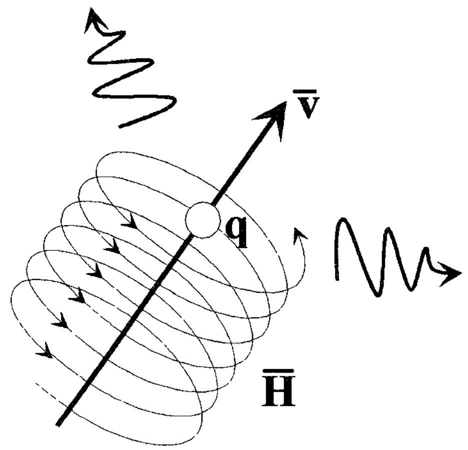
$$k_\pm = \omega\left[\sqrt{\mu\varepsilon - \left(\frac{\gamma+\beta}{2}\right)^2} \pm j\left(\frac{\gamma-\beta}{2}\right)\right] \tag{6}$$

where $\eta_\pm/k_\pm$ refers to the wave impedance/number of the RCP/LCP field components. Note that the chiral case results in $\eta_+ = \eta_-$. On the other hand, $\gamma = \beta$ results in $k_+ = k_-$, defining a class of materials as broad as the chiral reciprocal, and referred to as the nonactive case [9].

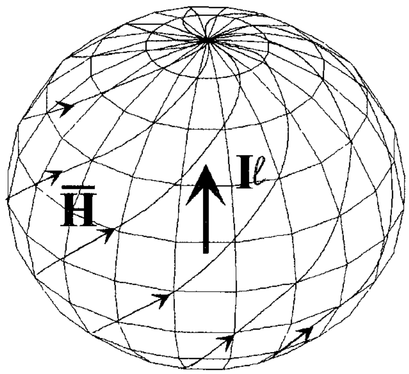
Unlike chiral materials, which do not admit linearly polarized solutions, nonactive materials do allow linearly polarized fields, leading to very interesting effects such as magnetic dipole fields, which do not close, but which are open spiral lines that go from pole to pole [9], or Cherenkov radiation with helical magnetic field lines [9]. Fig. 1 illustrate these points.

Basic equations pertinent to the partial fields have been presented and solved in [9] and will not be repeated here since they will not be employed.

[1] Thanks are extended to A. Lakhtakia for reference to the paper of Fel'd.



(a)



(b)

Fig. 1. Illustration of magnetic field lines in nonactive materials. (a) Single-cone Cherenkov radiation, helical lines. (b) Open spiral lines (going from pole to pole) of an elementary electric dipole. The figures describe the analytical representation contained in [9].

Once the above RCP/LCP decomposition is performed, each partial field obeys isotropic laws, i.e., the RCP fields sense a homogeneous space characterized by k_+ and η_+, whereas the LCP sense k_- and η_-. In absence of boundaries the RCP/LCP field components are uncoupled [9]. It then follows that any relationship that applies to RCP/LCP fields in homogeneous isotropic unbounded space is also applicable here for each field component. The effect on the total fields can then be obtained by reversing the previous process and expressing the partial fields in terms of the total fields. The inverse relationship being

$$\overline{H}_\pm = \frac{\eta_\mp \overline{H} \pm j\overline{E}}{\eta_+ + \eta_-}, \quad \overline{J}_\pm = \frac{j\eta_\mp \overline{J} \pm \overline{M}}{j(\eta_+ + \eta_-)}. \tag{7}$$

Our recent paper [7] introduced one such relationship. In [7, eq. (19)] it was found that partial RCP/LCP fields in unbounded domains do satisfy reciprocity, which can be cast in the form of two scalar equations

$$\int_\infty d\tau u_\pm = 0 \tag{8}$$

where

$$u_\pm = \overline{H}_\pm^{(2)} \cdot \overline{J}_\pm^{(1)} - \overline{H}_\pm^{(1)} \cdot \overline{J}_\pm^{(2)} \tag{9}$$

and the superscripts (1) and (2) refer to the two sets of sources $(\overline{J}^{(1)}, \overline{M}^{(1)})$ and $(\overline{J}^{(2)}, \overline{M}^{(2)})$. It may appear somewhat strange that we can obtain a relationship for the total fields/sources in the biisotropic medium out of (8), which applies to two different spaces (k_+, η_+ and k_-, η_-). The reason is that the two spaces are one and the same as seen by the sources. This can be seen from the network model of the excitation, which was presented graphically in [15, Fig. 2].

It should be noted that in reality (8) and (9) are a direct statement of reciprocity of RCP/LCP field components in biisotropic media.

Appropriate enforcement of (8) via (7) and (9) will yield two independent reciprocity theorems. Care must be exercised so as to identify components characteristic of the Lorentz theorem, or the newer, cross-polarized theorem [7]. Use of (7) in (9) results in (10), shown at the bottom of the page.

A. The First Relationship

To obtain a first theorem, we enforce according to (8)

$$\int_\infty d\tau j(\eta_+ + \eta_-)\left\{\frac{u_+}{\eta_-} + \frac{u_-}{\eta_+}\right\} = 0. \tag{11}$$

After some algebra, and using the following identity [9]

$$\eta_+\eta_- = \mu/\varepsilon \equiv \eta^2 \tag{12}$$

the above results in

$$\int_\infty d\tau\{\overline{M}^{(1)} \cdot \overline{E}^{(2)} + \eta^2 \overline{J}^{(1)} \cdot \overline{H}^{(2)}\} = \int_\infty d\tau\{\overline{M}^{(2)} \cdot \overline{E}^{(1)} + \eta^2 \overline{J}^{(2)} \cdot \overline{H}^{(1)}\} \tag{13}$$

which is identical in form to the new reciprocity theorem for isotropic media [7], and consequently we adopt the same abbreviation, namely

$$[1, 2] = [2, 1]. \tag{14}$$

This new reciprocity theorem is eminently cross-polarized and applies to all biisotropic media, including chiral as well as isotropic media, with no restrictions. To avoid confusion, it should be emphasized that all sources act on the same biisotropic space. We should also add that in a sense (13) and (14) is more general than the Lorentz theorem, since the latter clearly cannot apply to nonreciprocal (in the standard sense) biisotropic media (among the reasons, it cannot account for cross-polarization).

B. The Second Relationship

Use of (13) in the two relations (8), and after some elementary algebra, can be shown to lead to just a single integral relationship

$$\int_\infty d\tau\Big\{\left(\frac{\gamma + \beta}{\varepsilon}\right)[\overline{J}^{(1)} \cdot \overline{H}^{(2)} - \overline{J}^{(2)} \cdot \overline{H}^{(1)}] + \{\overline{E}^{(2)} \cdot \overline{J}^{(1)} - \overline{E}^{(1)} \cdot \overline{J}^{(2)} + \overline{M}^{(2)} \cdot \overline{H}^{(1)} - \overline{M}^{(1)} \cdot \overline{H}^{(2)}\}\Big\} = 0. \tag{15}$$

$$u_\pm = \frac{1}{j(\eta_+ + \eta_-)}\left\{\begin{array}{l} \pm\eta_\mp[\overline{H}^{(2)} \cdot \overline{M}^{(1)} - \overline{H}^{(1)} \cdot \overline{M}^{(2)}] + j\eta_\mp^2[\overline{H}^{(2)} \cdot \overline{J}^{(1)} - \overline{H}^{(1)} \cdot \overline{J}^{(2)}] \\ +j[\overline{E}^{(2)} \cdot \overline{M}^{(1)} - \overline{E}^{(1)} \cdot \overline{M}^{(2)}] \mp \eta_\mp[\overline{E}^{(2)} \cdot \overline{J}^{(1)} - \overline{E}^{(1)} \cdot \overline{J}^{(2)}] \end{array}\right\}. \tag{10}$$

By properly combining (15) with (13), we can write the result in a more symmetrical form, namely

$$\int_\infty d\tau\{\overline{\mathcal{J}}^{(1)} \cdot \overline{E}^{(2)} - \overline{\mathcal{M}}^{(1)} \cdot \overline{H}^{(2)}\} = \int_\infty d\tau\{\overline{\mathcal{J}}^{(2)} \cdot \overline{E}^{(1)} - \overline{\mathcal{M}}^{(2)} \cdot \overline{H}^{(1)}\} \tag{16}$$

which involves the real fields $(\overline{E}, \overline{H})$ and the *generalized* sources $(\overline{\mathcal{J}}, \overline{\mathcal{M}})$, which are defined according to

$$\overline{\mathcal{J}} = \overline{J} - \left(\frac{\gamma+\beta}{2\mu}\right)\overline{M}, \quad \overline{\mathcal{M}} = \overline{M} - \left(\frac{\gamma+\beta}{2\varepsilon}\right)\overline{J}. \tag{17}$$

Equation (16) can be rewritten in still another symmetrical form involving the real sources $(\overline{J}, \overline{M})$ and the *generalized* fields $(\overline{\mathcal{E}}, \overline{\mathcal{H}})$, defined according to

$$\overline{\mathcal{E}} = \overline{E} + \left(\frac{\gamma+\beta}{2\varepsilon}\right)\overline{H}, \quad \overline{\mathcal{H}} = \overline{H} + \left(\frac{\gamma+\beta}{2\mu}\right)\overline{E} \tag{18}$$

and which yield

$$\int_\infty d\tau\{\overline{J}^{(1)} \cdot \overline{\mathcal{E}}^{(2)} - \overline{M}^{(1)} \cdot \overline{\mathcal{H}}^{(2)}\} = \int_\infty d\tau\{\overline{J}^{(2)} \cdot \overline{\mathcal{E}}^{(1)} - \overline{M}^{(2)} \cdot \overline{\mathcal{H}}^{(1)}\}. \tag{19}$$

Equations (16) and (19) are different forms of the same identity, which is eminently co-polarized and constitutes the second reciprocal relationship for biisotropic media. For convenience we introduce the notation

$$\{1,2\} = \{2,1\} \tag{20}$$

to represent reciprocity in the sense of (16) or (19). Once again we emphasize that all sources act on the same biisotropic space. It should be noted that for the chiral and isotropic cases $\gamma + \beta = 0$, and the generalized quantities become the real quantities, and (20) reduces to the standard Lorentz Reciprocity Theorem, which is usually abbreviated as $\langle 1,2 \rangle = \langle 2,1 \rangle$. We should add that evidently (16) and (19) are generalizations of the Lorentz theorem.

III. Conclusion

Two generalized reciprocity theorems are presented that apply to homogeneous biisotropic media and do not invoke a complementary space. One of the theorems is eminently crosspolarized and constitutes the extension of a recently derived new theorem for isotropic regions. The second theorem is a generalization of the Lorentz theorem and is therefore eminently copolarized, invoking generalized sources or fields. The relationships presented here are useful not only for validation purposed of theory/numerical codes, but also because they constitute the foundation for new variational expressions leading to a reaction type development with capabilities to handle biisotropic/nonactive/chiral/isotropic materials.

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